Relative velocity and relative acceleration induced by the torsion in (L_n, g)- and U_n-spaces

S. Manoff
Bulgarian Academy of Sciences
Institute for Nuclear Research and Nuclear Energy
Department of Theoretical Physics
Blvd. Tzarigradsko Chaussee 72
1784 Sofia - Bulgaria

E-mail address: smanov@inrne.bas.bg

Abstract

The influence of the torsion on the relative velocity and on the relative acceleration between particles (points) in spaces with an affine connection and a metric [(L_n, g)-spaces] and in (pseudo) Riemannian spaces with torsion (U_n-spaces) is considered. Necessary and sufficient conditions as well as only necessary and only sufficient conditions for vanishing deformation, shear, rotation and expansion are found. The notion of relative acceleration and the related to it notions of shear, rotation and expansion accelerations induced by the torsion are determined. It is shown that the kinematic characteristics induced by the torsion (shear acceleration, rotation acceleration and expansion acceleration) could play the same role as the kinematic characteristics induced by the curvature and can (under given conditions) compensate their action as well as the action of external forces. The change of the rate of change of the length of a deviation vector field is given in explicit form for (L_n, g)- and U_n-spaces.

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1 Introduction

1.1 Differential geometry and space-time geometry

In the last years, the evolution of the relations between differential geometry and space-time geometry has made some important steps toward applications of more comprehensive differential-geometric structures in the models of space-time than these used in (pseudo) Riemannian spaces without torsion (V_n-spaces).

1. It has been proved recently that every differentiable manifold with one affine connection and metrics [(L_n, g)-space] [1] could be used as a model for a space-time. In it the equivalence principle (related to the vanishing of the components of an affine connection at a point or on a curve in a manifold) holds [2] $\div$ [8], [9]. Even if the manifold has two different (not only by sign) connections for tangent and co-tangent vector fields [$(\overline{L}_n, g)$-space] [10] [11] the principle of equivalence is fulfilled at least for one of the two types of vector fields [12]. On this grounds, every free moving spinless test particle in a suitable basic system (frame of reference) [13] [14] will fulfil an equation identical with the equation for a free moving spinless test particle in the Newtons mechanics or in the special relativity. In $(\overline{L}_n, g)$- and (L_n, g)-spaces, this equation could be the autoparallel equation [different from the geodesic equation in contrast to the case of (pseudo) Riemannian spaces without torsion (V_n-spaces)].

2. There are evidences that (L_n, g)- and $(\overline{L}_n, g)$-spaces can have similar structures as the V_n-spaces for describing dynamical systems and the gravitational interaction. In such type of spaces one could use Fermi-Walker transports [15] [16] [17] conformal transports [18] [19] and different frames of reference [14]. All these notions appear as generalizations of the corresponding notions in V_n-spaces. For instance, in (L_n, g)- and $(\overline{L}_n, g)$-spaces a proper non-rotating accelerated observer's frame of reference could be introduced by analogy of the same type of frame of reference related to a Fermi-Walker transport [20] in the Einstein theory of gravitation (ETG).

3. All kinematic characteristics related to the notion of relative velocity [21] as well as the kinematic characteristics related to the notion of relative acceleration have been worked out for (L_n, g)- and $(\overline{L}_n, g)$-spaces without changing their physical interpretation in V_n-spaces [22]. Necessary and sufficient conditions as well as only necessary an only sufficient conditions for vanishing shear, rotation and expansion acceleration induced by the curva-

ture are found [22]. The last results are related to the possibility of building a theoretical basis for construction of gravitational wave detectors on the grounds of gravitational theories over (L_n, g) and $(\overline{L}_n, g)$–spaces.

Usually, in the gravitational experiments the measurements of two basic objects are considered [23]:

(a) The relative velocity between two particles (or points) related to the rate of change of the length (distance) between them. The change of the distance is supposed to be caused by the gravitational interaction.

(b) The relative accelerations between two test particles (or points) of a continuous media. These accelerations are related to the curvature of the space-time and supposed to be induced by a gravitational force.

Together with the accelerations induced by the curvature in V_n-spaces accelerations induced by the torsion would appear in U_n-spaces, as well as by torsion and by non-metricity in (L_n, g)- and $(\overline{L}_n, g)$-spaces.

In particular, in other models of a space-time [different from the (pseudo) Riemannian spaces without torsion] the torsion has to be taken into account if we consider the characteristics of the space-time.

4. On the one hand, until now, there are a few facts that the torsion could induce some very small and unmesasurable effects in quantum mechanical systems considered in spaces with torsion [24]. At the same time, there are no evidences that the model's descriptions of interactions on macro level should include the torsion as a necessary mathematical tool. From this (physical) point of view the influence of the torsion on dynamical systems could be ignored since it could not play an important role in the description of physical systems in the theoretical gravitational physics. On the other hand, from mathematical point of view (as we will try to show in this paper), the role of the torsion in new theories for description of dynamical systems could be important and not ignorable.

1.2 Problems and results

In this paper the influence of the torsion on the relative velocity and the relative acceleration between particles moving in (L_n, g)- and U_n-spaces ($n = 4$) is considered.

In Sec. 2. the notion of relative velocity as well as the related to it notions of shear, rotation and expansion velocities, induced by the torsion, are determined. The change of the length of a vector field (deviation vector field) related to two moving particles is given in an explicit form for (L_n, g)- and

U_n-spaces. It is shown that the existence of deformation, shear, rotation and expansion (velocities), induced by the torsion, can compensate (under given conditions) the action of these induced by external forces. The necessary and sufficient conditions as well as only necessary and only sufficient conditions for vanishing deformation, shear, rotation and expansion are found. The vanishing of all these quantities could be caused by the torsion.

In Sec. 3. the notion of relative acceleration and the related to it notions of shear, rotation and expansion accelerations induced by the torsion are found. It is shown that the existence of kinematic characteristics, induced by the torsion, can compensate (under given conditions) the action of these induced by the curvature or by external forces. The change of the rate of change of the length of a deviation vector field is given in explicit form for (L_n, g)- and U_n-spaces.

Concluding remarks comprise the final Sec. 4. The most considerations are given in details (even in full details) for those readers who are not familiar with the considered problems.

2 Relative velocity and its kinematic characteristics induced by the torsion

Let us now recall some well known facts from differential geometry. Every contravariant vector field $\xi \in T(M)$ over a differentiable manifold M can be written by means of its projection along and orthogonal to a contravariant non-null (nonisotropic) vector field u in two parts - one collinear to u and one - orthogonal to u, i.e.

$$\xi = \frac{l}{e}.u + h^u[g(\xi)] \ = \frac{l}{e}.u + \overline{g}[h_u(\xi)] \ , \tag{1}$$

where $\overline{g}[h_u(\xi)] := \xi_\perp = g^{ik} \cdot h_{kl} \cdot \xi^l \cdot e_i$ is called deviation vector field and

$$\begin{gathered}
l = g(\xi, u) = g_{ij} \cdot \xi^i \cdot u^j \ , \\
h^u = \overline{g} - \tfrac{1}{e}.u \otimes u \ , \ \xi = \xi^i.\partial_i = \xi^k.e_k \ , \qquad h^u = h^{ij}.e_i.e_j \ , \\
\overline{g}(h_u)\overline{g} = h^u \ , \quad h_u(\overline{g})(g) = h_u \ , \quad h^u(g)(\overline{g}) = h^u \ , \quad g(h^u)g = h_u \ , \\
h_u = g - \tfrac{1}{e} \cdot g(u) \otimes g(u) \ , \\
\overline{g} = g^{ij} \cdot e_i.e_j \ , \quad e_i.e_j = \tfrac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) \ , \ g^{ij} = g^{ji} \ , \\
g = g_{ij} \cdot e^i.e^j \ , \ e^i.e^j = \tfrac{1}{2}(e^i \otimes e^j + e^j \otimes e^i) \ , \ g_{ij} = g_{ji} \ .
\end{gathered} \tag{2}$$

Therefore, the covariant derivative $\nabla_u \xi$ of the contravariant vector field ξ along a non-null vector field u [as a result of the action of the contravariant differential operator $\nabla_u : \xi \to \nabla_u \xi \in T(M)$, $\nabla_u \xi = \xi^i{}_{;j} \cdot u^j \cdot \partial_i$, $\xi^i{}_{;j} = \xi^i{}_{,j} + \Gamma^i_{lj} \cdot \xi^l$, Γ^i_{lj} are the components of the affine connection Γ in a (L_n, g)-space] can be written in the form

$$\nabla_u \xi = \frac{\bar{l}}{e} \cdot u + \overline{g}[h_u(\nabla_u \xi)] \ , \qquad \bar{l} = g(\nabla_u \xi, u) \ . \tag{3}$$

2.1 Relative velocity in (L_n, g)-spaces

The notion *relative velocity* vector field (relative velocity) ${}_{rel}v$ can be defined in (L_n, g)-spaces (regardless of its physical interpretation) as the projection [orthogonal to a non-null (nonisotropic) contravariant vector field u] of the first covariant derivative $\nabla_u \xi$ (along the same non-null vector field u) of (another) contravariant vector field ξ, i.e.

$${}_{rel}v = \overline{g}(h_u(\nabla_u \xi)) = g^{ij} \cdot h_{jk} \cdot \xi^k{}_{;l} \cdot u^l \cdot e_i \ , \quad e_i = \partial_i \text{ (in a co-ordinate basis)}, \tag{4}$$

where (the indices in a co-ordinate and in a non-co-ordinate basis are written in both cases as Latin indices instead of Latin and Greek indices)

$$\begin{gathered} h_u = g - \tfrac{1}{e}.g(u) \otimes g(u) \ , \ h_u = h_{ij} \cdot e^i.e^j \ , \ \overline{g} = g^{ij} \cdot e_i.e_j , \\ \nabla_u \xi = \xi^i{}_{;j} \cdot u^j \cdot e_i \ , \qquad \xi^i{}_{;j} = e_j \xi^i + \Gamma^i_{kj} \cdot \xi^k , \qquad \Gamma^i_{kj} \neq \Gamma^i_{jk} \ , \\ e = g(u, u) = g_{ij} \cdot u^i \cdot u^j = u_i \cdot u^i \neq 0 \ , \qquad g(u) = g_{ik} \cdot u^k \cdot e^i = u_i \cdot e^i \ , \\ h_u(\nabla_u \xi) = h_{ij} \cdot \xi^j{}_{;k} \cdot u^k \cdot e^i \ , \qquad h_{ij} = g_{ij} - \tfrac{1}{e} \cdot u_i \cdot u_j \ , \\ h_u(\nabla_u \xi) = h_{ij} \cdot \xi^j{}_{;k} \cdot u^k \cdot e^i \ . \end{gathered} \tag{5}$$

In a co-ordinate basis $e_j \xi^i = \xi^i{}_{,j} = \partial_j \xi^i = \partial \xi^i / \partial x^j$, $e^j = dx^j$, $e_i = \partial_i = \partial / \partial x^i$, $u = u^i \cdot \partial_i, e_{,k} = e_k e = \partial_k e$.

Using the relation [25] between the Lie derivative $\pounds_\xi u$ and the covariant derivative $\nabla_\xi u$

$$\pounds_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi, u) \ , \qquad T(\xi, u) = T_{ij}{}^k.\xi^i.u^j.e_k \ , \tag{6}$$

one can write $\nabla_u \xi$ in the form

$$\nabla_u \xi = (k) g(\xi) - \pounds_\xi u \ = k[g(\xi)] - \pounds_\xi u, \tag{7}$$

or, taking into account the above expression for ξ, in the form $\nabla_u \xi = k[h_u(\xi)] + \frac{l}{e}.a - \pounds_\xi u$, where

$$k[g(\xi)] = \nabla_\xi u - T(\xi, u) \ , \quad k = (u^i{}_{;l} - T_{lk}{}^i . u^k).g^{lj}.e_i \otimes e_j = k^{ij}.e_i \otimes e_j \ ,$$
$$k[g(u)] = k(g)u = k^{ij}.g_{jk}.u^k.e_i = a = \nabla_u u = u^i{}_{;j}.u^j.e_i \ . \tag{8}$$

$T_{ij}{}^k$ are the components of the torsion tensor T:

$$T_{ij}{}^k = -T_{ji}{}^k = \Gamma^k_{ji} - \Gamma^k_{ij} - C_{ij}{}^k \quad \text{(in a non-co-ordinate basis } \{e_k\}) \ ,$$
$$[e_i, e_j] = \pounds_{e_i} e_j \ = C_{ij}{}^k . e_k \ ,$$
$$T_{ij}{}^k = \Gamma^k_{ji} - \Gamma^k_{ij} \quad \text{(in a co-ordinate basis } \{\partial_k\} \ ) \ ,$$

For $h_u(\nabla_u \xi)$, it follows

$$h_u(\nabla_u \xi) = h_u(\frac{l}{e} \cdot a - \pounds_\xi u) + h_u(k) h_u(\xi) \ , \tag{9}$$

where $h_u(k)h_u(\xi) = h_{ik} \cdot k^{kl} \cdot h_{lj} \cdot \xi^j \cdot e^i$, $h_u(u) = 0$, $u(h_u) = 0$, $h_u(k)h_u(u) = 0$, $(u)h_u(k)h_u = 0$.

If we introduce the abbreviation

$$d = h_u(k)h_u = h_{ik} \cdot k^{kl} \cdot h_{lj} \cdot e^i \otimes e^j = d_{ij} \cdot e^i \otimes e^j \ , \tag{10}$$

the expression for ${}_{rel}v$ can take the form

$${}_{rel}v = \overline{g}[h_u(\nabla_u \xi)] = \overline{g}(h_u)(\frac{l}{e} \cdot a - \pounds_\xi u) + \overline{g}[d(\xi)] =$$

$$= [g^{ik} \cdot h_{kl} \cdot (\frac{l}{e} \cdot a^l - \pounds_\xi u^l) + g^{ik} \cdot d_{kl} \cdot \xi^l] \cdot e_i \ = {}_{rel}v^i \cdot e_i \ , \tag{11}$$

or

$$g({}_{rel}v) = h_u(\nabla_u \xi) = h_u(\frac{l}{e} \cdot a - \pounds_\xi u) + d(\xi) \ . \tag{12}$$

For the special case, when the vector field ξ is orthogonal to u, i.e. $\xi = \overline{g}[h_u(\xi)]$, and the Lie derivative of u along ξ is zero, i.e. $\pounds_\xi u = 0$, then the relative velocity can be written in the form $g({}_{rel}v) = d(\xi)$ or in the form

$${}_{rel}v = \overline{g}[d(\xi)].$$

Remark. All further calculations leading to a useful representation of d are quite straightforward. The problem here was the finding out a representation of $h_u(\nabla_u \xi)$ in the form (9) which is not a trivial task.

2.2 Deformation velocity, shear velocity, rotation velocity and expansion velocity

The covariant tensor field d is a generalization for (L_n, g)-spaces of the well known *deformation velocity* tensor for V_n-spaces [21] , [26] . It is usually represented by means of its three parts: the trace-free symmetric part, called *shear velocity* tensor (shear), the anti-symmetric part, called *rotation velocity* tensor (rotation) and the trace part, in which the trace is called *expansion velocity* (expansion) invariant.

After some more complicated as for V_n-spaces calculations, the deformation velocity tensor d can be given in the form

$$d = h_u(k)h_u = h_u(k_s)h_u + h_u(k_a)h_u = \sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u \ , \tag{13}$$

where $k_s = {}_sk^{ij} \cdot e_i.e_j$, ${}_sk^{ij} = \frac{1}{2}(k^{ij} + k^{ji})$, ${}_ak = {}_ak^{ij} \cdot e_i \wedge e_j$, ${}_ak^{ij} = \frac{1}{2}(k^{ij} - k^{ji})$, $e_i \wedge e_j = \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)$

The tensor σ is the *shear velocity* tensor (shear) ,

$$\begin{array}{c} \sigma = {}_sE - {}_sP = E - P - \frac{1}{n-1} \cdot \overline{g}[E-P] \cdot h_u = \sigma_{ij} \cdot e^i.e^j = \\ = E - P - \frac{1}{n-1} \cdot (\theta_o - \theta_1) \cdot h_u \ , \end{array} \tag{14}$$

where

$$\begin{array}{c} {}_sE = E - \frac{1}{n-1} \cdot \overline{g}[E] \cdot h_u \ , \qquad \overline{g}[E] = g^{ij} \cdot E_{ij} = \theta_o \ , \\ E = h_u(\varepsilon)h_u \ , \qquad k_s = \varepsilon - m \ , \qquad \varepsilon = \frac{1}{2}(u^i{}_{;l} \cdot g^{lj} + u^j{}_{;l} \cdot g^{li}) \cdot e_i.e_j \ , \\ m = \frac{1}{2}(T_{lk}{}^i \cdot u^k \cdot g^{lj} + T_{lk}{}^j \cdot u^k \cdot g^{li}) \cdot e_i.e_j \ . \end{array} \tag{15}$$

The tensor ${}_sE$ is the *torsion-free shear velocity* tensor, ${}_sP$ is the *shear velocity* tensor *induced by the torsion*,

$$\begin{array}{c} {}_sP = P - \frac{1}{n-1} \cdot \overline{g}[P] \cdot h_u \ , \qquad \overline{g}[P] = g^{kl} \cdot P_{kl} = \theta_1, \quad P = h_u(m)h_u \ , \\ \theta_1 = T_{kl}{}^k \cdot u^l \ , \quad \theta_o = u^n{}_{;n} - \frac{1}{2e}(e_{,k} \cdot u^k - g_{kl;m} \cdot u^m \cdot u^k \cdot u^l) \ , \ \theta = \theta_o - \theta_1 \ . \end{array} \tag{16}$$

The invariant θ is the *expansion velocity,* θ_o is the *torsion-free expansion velocity,* θ_1 is the *expansion velocity induced by the torsion.*

The tensor ω is the *rotation velocity* tensor (rotation velocity),

$$\omega = h_u(k_a)h_u = h_u(s)h_u - h_u(q)h_u = S - Q \ ,$$
$$s = \frac{1}{2}(u^k{}_{;m} \cdot g^{ml} - u^l{}_{;m} \cdot g^{mk}) \cdot e_k \wedge e_l \ ,$$
$$q = \frac{1}{2}(T_{mn}{}^k \cdot g^{ml} - T_{mn}{}^l \cdot g^{mk}) \cdot u^n \cdot e_k \wedge e_l \ , \quad S = h_u(s)h_u \ , \quad Q = h_u(q)h_u \ . \tag{17}$$

The tensor S is the *torsion-free rotation velocity* tensor, Q is the *rotation velocity* tensor *induced by the torsion.*

By means of the expressions for σ, ω and θ the deformation velocity tensor d can be decomposed in two parts: d_0 and d_1

$$d = d_o - d_1 \ , \quad d_o = {}_sE + S + \frac{1}{n-1}.\theta_o.h_u \ , \quad d_1 = {}_sP + Q + \frac{1}{n-1}.\theta_1.h_u \ , \tag{18}$$

where d_o is the *torsion-free deformation velocity* tensor and d_1 is the *deformation velocity* tensor *induced by the torsion.* For the case of V_n-spaces $d_1 = 0$ (${}_sP = 0$, $Q = 0$, $\theta_1 = 0$).

Remark. The shear velocity tensor σ and the expansion velocity θ can also be written in the forms

$$\sigma = \frac{1}{2}\{h_u(\nabla_u\overline{g} - \pounds_u\overline{g})h_u - \frac{1}{n-1} \cdot (h_u[\nabla_u\overline{g} - \pounds_u\overline{g}]) \cdot h_u\} =$$
$$= \frac{1}{2}\{h_{ik} \cdot (g^{kl}{}_{;m} \cdot u^m - \pounds_u g^{kl}) \cdot h_{lj} - \frac{1}{n-1} \cdot h_{kl} \cdot (g^{kl}{}_{;m} \cdot u^m - \pounds_u g^{kl}) \cdot h_{ij}\} \cdot e^i.e^j \ , \tag{19}$$
$$\theta = \frac{1}{2} \cdot h_u[\nabla_u\overline{g} - \pounds_u\overline{g}] = \frac{1}{2}h_{ij} \cdot (g^{ij}{}_{;k} \cdot u^k - \pounds_u g^{ij}) \ .$$

The physical interpretation of the velocity tensors d, σ, ω, and of the invariant θ for the case of V_4-spaces [27] , [28] , can also be extended for (L_4, g)-spaces. In this case the torsion plays an equivalent role in the velocity tensors as the covariant derivative. It is easy to be seen that the existence of some kinematic characteristics (${}_sP$, Q, θ_1) depends on the existence of the torsion tensor field. They vanish if it is equal to zero (e.g. in V_n-spaces). On the other side, *the kinematic characteristics, induced by the torsion, can compensate the result of the action of the torsion-free kinematic characteristics.* If $d = 0$, $\sigma = 0$, $\omega = 0$, $\theta = 0$, then we could have the relations $d_0 = d_1$, ${}_sE = {}_sP$, $S = Q$, $\theta_0 = \theta_1$ respectively leading to vanishing the relative velocity ${}_{rel}v$ under the additional conditions $g(u, \xi) = l = 0$ and $\pounds_\xi u = 0$. Since $\nabla_u \xi = \frac{\overline{l}}{e} \cdot u + {}_{rel}v$ and

$\bar{l} = g(\nabla_u \xi, u) = ul - g(\xi, a) - (\nabla_u g)(\xi, u) = -g(\xi, a) - (\nabla_u g)(\xi, u)$ for $l = 0$, $\nabla_u \xi = -(\nabla_u g)(\xi, u)$ can be seen under the above conditions as a measure for the non-metricity $(\nabla_u g)$ of the space-time if $a = 0$ and ${}_{rel}v = 0$.

2.3 Special contravariant vector fields with vanishing kinematic characteristics related to the relative velocity

The explicit forms of the quantities d, σ, ω, and θ related to the relative velocity can be used for finding conditions for existence of special types of contravariant vector fields with vanishing characteristics induced by the relative velocity.

2.3.1 Contravariant vector fields with vanishing deformation velocity ($d = 0$)

If we consider the explicit form for d

$$d := h_u(k)h_u$$

we can prove the following propositions:

Proposition 1. The necessary and sufficient condition for the existence of a non-null contravariant vector field u with vanishing deformation velocity ($d = 0$) is the condition

$$k = \frac{1}{e} \cdot \{a \otimes u + u \otimes [g(u)](k)\} - \frac{1}{2e^2} \cdot [ue - (\nabla_u g)(u, u)] \cdot u \otimes u \ ,$$

or in a co-ordinate basis

$$\begin{aligned} k^{ij} \ &= \ \frac{1}{e} \cdot (a^i \cdot u^j + u^i \cdot k^{lj} \cdot g_{lm} \cdot u^m) \\ &\quad -\frac{1}{2e^2} \cdot (e_{,k} \cdot u^k - g_{kl;m} \cdot u^m \cdot u^k \cdot u^l) \cdot u^i \cdot u^j \ . \end{aligned}$$

Proof. 1. Necessity. From $d = h_u(k)h_u$, after writing the explicit form of h_u, it follows that

$$\begin{aligned} d \ &= \ g(k)g - \frac{1}{e} \cdot g(u) \otimes [g(u)](k)g - \frac{1}{e} \cdot g(k)[g(u)] \otimes g(u) \\ &\quad +\frac{1}{e^2} \cdot [g(u)](k)[g(u)] \cdot u \otimes u \ . \end{aligned}$$

Since $(k)[g(u)] = a = \nabla_u u$, it follows further that

$$[g(u)](k)[g(u)] = [g(u)](a) = g(u,a) = \frac{1}{2} \cdot [ue - (\nabla_u g)(u,u)] \ .$$

Therefore,

$$\begin{aligned} d &= 0 : g(k)g = \frac{1}{e} \cdot \{g(u) \otimes [g(u)](k)g + g(a) \otimes g(u)\} \\ &\quad - \frac{1}{e^2} \cdot g(u,a) \cdot g(u) \otimes g(u) \ . \end{aligned}$$

From $\overline{g}(g(k)g)\overline{g} = k$, we obtain

$$\begin{aligned} g(k)g &= \frac{1}{e} \cdot \{a \otimes u + u \otimes [g(u)](k)\} \\ &\quad - \frac{1}{2 \cdot e^2} \cdot [ue - (\nabla_u g)(u,u)] \cdot u \otimes u \ . \end{aligned}$$

2. Sufficiency. From the explicit form of k, it follows that

$$\begin{aligned} g(k)g &= \frac{1}{e} \cdot g(u) \otimes [g(u)](k)g + \frac{1}{e} \cdot g(k)[g(u)] \otimes g(u) \\ &\quad - \frac{1}{e^2} \cdot g(u,a) \cdot g(u) \otimes g(u) \end{aligned}$$

which is identical to $h_u(k)h_u = d = 0$.

Special case: $\nabla_u u = a := 0$, $\nabla_\xi g := 0$ for $\forall \xi \in T(M)$ (U_n-space), $ue = 0 : e = \text{const.} \neq 0$ (u is normalized, non-null contravariant vector field).

$$d = 0 : k = \frac{1}{e} \cdot u \otimes [g(u)](k) \ ,$$

$$(k)[g(\xi)] = \frac{1}{e} \cdot u \otimes [g(u)](k)[g(\xi)] = \frac{1}{e} \cdot [g(u)](k)[g(\xi)] \cdot u \ ,$$

$$\begin{aligned} (k)[g(\xi)] &= (u^i{}_{;l} - T_{lk}{}^i \cdot u^k) \cdot g^{lm} \cdot g_{mj} \cdot \xi^j \cdot \partial_i = \nabla_\xi u - T(\xi, u) = \\ &= \nabla_u \xi - \pounds_\xi u \ , \end{aligned}$$

$$_{rel}v = \overline{g}[h_u(\nabla_u \xi)] = -\overline{g}(h_u)(\pounds_\xi u) \text{ for } \forall \xi \in T(M) \ . \tag{20}$$

Proposition 2. A sufficient condition for the existence of a non-null contravariant vector field with vanishing deformation velocity ($d = 0$) is the condition

$$k = 0 \ ,$$

equivalent to the condition

$$\nabla_\xi u = T(\xi, u) \quad \text{for } \forall \xi \in T(M) \ ,$$

or in a co-ordinate basis

$$k^{ij} = 0 : u^i{}_{;j} = T_{jk}{}^i \cdot u^k \ .$$

Proof. From $k = 0$ and $(k)[g(\xi)] = \nabla_\xi u - T(\xi, u)$ for $\forall \xi \in T(M)$, it follows that $\nabla_\xi u - T(\xi, u) = 0$ or in a co-ordinate basis $u^i{}_{;j} - T_{jk}{}^i \cdot u^k = 0$. In this case $\pounds_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi, u) = -\nabla_u \xi$.

Corollary. A deformation free contravariant vector field u with $k = 0$ is an auto-parallel contravariant vector field.

Proof. It follows immediately from the condition $\nabla_\xi u = T(\xi, u)$ and for $\xi = u$ that $\nabla_u u = a = 0$.

Proposition 3. The necessary condition for the existence of a deformation free contravariant vector field u with $k = 0$ is the condition

$$[R(u, v)]\xi = [\pounds\Gamma(u, v)]\xi \qquad \text{for } \ \forall \xi, v \in T(M) \ ,$$

or in a co-ordinate basis

$$R^k{}_{ilj} \cdot u^l = \pounds_u \Gamma^k_{ij} \ .$$

Proof. By the use of the explicit form of the curvature operator $R(u, v)$ acting on a contravariant vector field ξ

$$[R(u, v)]\xi = \nabla_u \nabla_v \xi - \nabla_v \nabla_u \xi - \nabla_{\pounds_u v} \xi \ , \qquad \xi,\ v,\ u \in T(M) \ ,$$

and the explicit form of the deviation operator $\pounds\Gamma(u, v)$ acting on a contravariant vector field ξ

$$[\pounds\Gamma(u, v)]\xi = \pounds_u \nabla_v \xi - \nabla_v \pounds_u \xi - \nabla_{\pounds_u v} \xi$$

we obtain under the condition $\nabla_\xi u = T(\xi, u)$ (equivalent to the condition $\pounds_u \xi = \nabla_u \xi$)

$$\begin{aligned}
[\pounds\Gamma(u, v)]\xi &= \pounds_u \nabla_v \xi - \nabla_v \pounds_u \xi - \nabla_{\pounds_u v} \xi = \\
&= \nabla_u \nabla_v \xi - \nabla_{\nabla_v \xi} u - T(u, \nabla_v \xi) - \nabla_v \nabla_u \xi - \nabla_{\pounds_u v} \xi = \\
&= \nabla_u \nabla_v \xi - \nabla_v \nabla_u \xi - \nabla_{\pounds_u v} \xi - [\nabla_{\nabla_v \xi} u + T(u, \nabla_v \xi)] = \\
&= [R(u, v)]\xi - [\nabla_{\nabla_v \xi} u + T(u, \nabla_v \xi)] \ .
\end{aligned}$$

Since
$$\nabla_{\nabla_v \xi} u + T(u, \nabla_v \xi) = 0 \quad \text{ for } \quad \forall v, \xi \in T(M) \ ,$$
we have
$$[R(u, v)]\xi = [\pounds\Gamma(u, v)]\xi \quad \text{ for } \quad \forall v, \xi \in T(M) \ \ .$$

The last condition appears as the integrability condition for the equation for u
$$\nabla_\xi u = T(\xi, u) \quad \text{for} \quad \forall \xi \in T(M) \ \ .$$

Proposition 4. A deformation-free contravariant non-null vector field u with $k = 0$ is an auto-parallel non-null shear-free ($\sigma = 0$), rotation-free ($\omega = 0$) and expansion-free ($\theta = 0$) contravariant vector field with vanishing deformation acceleration ($A = 0$) [22].

Proof. If $k = k^{ij} \cdot \partial_i \otimes \partial_j = 0$ and $\nabla_u u = a = 0$, then $k_s = k^{(ij)} \cdot \partial_i . \partial_j = \frac{1}{2} \cdot (k^{ij} + k^{ji}) \cdot \partial_i . \partial_j = 0$, and $k_a = k^{[ij]} \cdot \partial_i \wedge \partial_j = \frac{1}{2} \cdot (k^{ij} - k^{ji}) \cdot \partial_i \wedge \partial_j = 0$. Therefore, $\sigma = h_u(k_s)h_u - \frac{1}{n-1} \cdot \overline{g}[h_u(k_s)h_u] \cdot h_u = 0$, $\theta = \overline{g}[h_u(k_s)h_u] = 0$, and $\omega = h_u(k_a)h_u = 0$. From the explicit form of the deformation acceleration A, it follows that $A = 0$.

From the identity for the Riemannian tensor $R^i{}_{jkl}$
$$\begin{aligned} R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj} \ \equiv \ & T_{jk}{}^i{}_{;l} + T_{lj}{}^i{}_{;k} + T_{kl}{}^i{}_{;j} + \\ & + T_{jk}{}^m \cdot T_{ml}{}^i + T_{lj}{}^m \cdot T_{mk}{}^i + T_{kl}{}^m \cdot T_{mj}{}^i \ , \end{aligned}$$
after contraction with g^l_i and summation over l we obtain
$$\begin{aligned} R_{jk} - R_{kj} + R^i{}_{ijk} \ \equiv \ & T_{jk}{}^i{}_{;i} + T_{ij}{}^i{}_{;k} - T_{ik}{}^i{}_{;j} + \\ & + T_{jk}{}^m \cdot T_{mi}{}^i + T_{ij}{}^m \cdot T_{mk}{}^i - T_{ik}{}^m \cdot T_{mj}{}^i \ . \end{aligned}$$

If we introduce the abbreviations
$${}_aR_{jk} := \frac{1}{2} \cdot (R_{jk} - R_{kj}) \ , \qquad T_{ji}{}^i := T_j \ ,$$
where $T_{ik}{}^i = -T_{ki}{}^i = -T_k$, then the last expression for R_{jk} can be written in the form
$$\begin{aligned} 2 \cdot {}_aR_{jk} \ \equiv \ & -R^i{}_{ijk} + T_{jk}{}^i{}_{;i} + T_{ij}{}^i{}_{;k} - T_{ik}{}^i{}_{;j} + \\ & + T_{jk}{}^m \cdot T_m + T_{ij}{}^m \cdot T_{mk}{}^i - T_{ik}{}^m \cdot T_{mj}{}^i \ . \end{aligned}$$

Therefore, ${}_aR_{ij} \cdot u^j$ can be written in the form

$$\begin{aligned} 2 \cdot {}_aR_{ij} \cdot u^j + R^i{}_{ijk} \cdot u^j &= T_{k;j} \cdot u^j - T_{j;k} \cdot u^j + T_{jk}{}^i{}_{;i} \cdot u^j + \\ &\quad + T_m \cdot T_{jk}{}^m \cdot u^j + T_{ij}{}^m \cdot u^j \cdot T_{mk}{}^i - T_{ik}{}^m \cdot T_{mj}{}^i \cdot u^j \ . \end{aligned}$$

From the other side, from $u^i{}_{;j} = T_{jl}{}^i \cdot u^l$ and $a^i = u^i{}_{;j} \cdot u^j = 0$, we have

$$u^i{}_{;j;k} = T_{jl}{}^i{}_{;k} \cdot u^l + T_{jm}{}^i \cdot T_{kl}{}^m \cdot u^l \ ,$$

$$\begin{aligned} u^i{}_{;j;k} - u^i{}_{;k;j} &= -u^l \cdot R^i{}_{ljk} + T_{jk}{}^m \cdot T_{ml}{}^i \cdot u^l = \\ &= T_{jl}{}^i{}_{;k} \cdot u^l + T_{jm}{}^i \cdot T_{kl}{}^m \cdot u^l - \\ &\quad - T_{kl}{}^i{}_{;j} \cdot u^l - T_{km}{}^i \cdot T_{jl}{}^m \cdot u^l \ , \end{aligned}$$

$$\begin{aligned} u^l \cdot R^i{}_{ljk} &= T_{jk}{}^m \cdot T_{ml}{}^i \cdot u^l + T_{kl}{}^i{}_{;j} \cdot u^l - T_{jl}{}^i{}_{;k} \cdot u^l + \\ &\quad + T_{km}{}^i \cdot T_{jl}{}^m \cdot u^l - T_{jm}{}^i \cdot T_{kl}{}^m \cdot u^l \ , \end{aligned}$$

$$R_{lj} \cdot u^l = -T_{l;j} \cdot u^l - T_{jl}{}^i{}_{;i} \cdot u^l - T_m \cdot T_{jl}{}^m \cdot u^l \ ,$$

$$R_{lj} \cdot u^l \cdot u^j = {}_sR_{lj} \cdot u^l \cdot u^j = I = -T_{l;j} \cdot u^l \cdot u^j = -(T_i \cdot u^i)_{;j} \cdot u^j = \dot{\theta}_1 \ .$$

By the use of the decompositions $R_{ij} = {}_aR_{ij} + {}_sR_{ij}$, $R_{ij} \cdot u^j = {}_aR_{ij} \cdot u^j + {}_sR_{ij} \cdot u^j$, and the above expression for $R_{lj} \cdot u^l$, we can find the following relations

$$2 \cdot {}_aR_{jk} \cdot u^j = T_{k;j} \cdot u^j - T_{j;k} \cdot u^j - 2 \cdot T_{kj}{}^i{}_{;i} \cdot u^j - 2 \cdot T_m \cdot T_{kj}{}^m \cdot u^j \ ,$$

$$R^i{}_{ijk} \cdot u^j = (T_{kj}{}^i{}_{;i} + T_m \cdot T_{kj}{}^m + T_{ij}{}^m \cdot T_{mk}{}^i - T_{ik}{}^m \cdot T_{mj}{}^i) \cdot u^j \ ,$$

$$2 \cdot {}_sR_{jk} \cdot u^j = -(T_{j;k} + T_{k;j}) \cdot u^j \ .$$

It follows that in a $(\overline{L}_n, g)$-space the projections of the symmetric part of the Ricci tensor on the non-null contravariant vector field u with $k = 0$ is depending on the covariant derivatives of T_i (respectively on the torsion $T_{ik}{}^l$) and not on the torsion $T_{ik}{}^l$ itself.

2.3.2 Contravariant non-null (nonisotropic) vector fields with vanishing shear velocity ($\sigma = 0$)

If we consider the explicit form of the shear velocity tensor (shear velocity, shear)

$$\sigma = h_u(k_s)h_u - \frac{1}{n-1} \cdot \overline{g}[h_u(k_s)h_u] \cdot h_u$$

we can prove the following propositions:

Proposition 5. The necessary and sufficient condition for the existence of a shear-free non-null contravariant vector field is the condition

$$\begin{aligned} k_s &= \frac{1}{2 \cdot e} \cdot \{u \otimes a + a \otimes u + u \otimes [g(u)](k) + [g(u)](k) \otimes u - \\ &\quad - \frac{1}{e} \cdot [ue - (\nabla_u g)(u,u)] \cdot u \otimes u\} + \\ &\quad + \frac{1}{n-1} \cdot \theta \cdot h^u , \end{aligned}$$

or in a co-ordinate basis

$$\begin{aligned} h_s^{ij} &= \frac{1}{2 \cdot e} \cdot \{u^i \cdot a^j + u^j \cdot a^i + u^i \cdot g_{\overline{mn}} \cdot u^n \cdot k^{mj} + u^j \cdot g_{\overline{mn}} \cdot u^n \cdot k^{mi} - \\ &\quad - \frac{1}{e} \cdot [e_{,k} \cdot u^k - g_{km;n} \cdot u^n \cdot u^{\overline{k}} \cdot u^{\overline{m}}] \cdot u^i \cdot u^j\} + \frac{1}{n-1} \cdot \theta \cdot h^{ij} . \end{aligned}$$

Proof. 1. Necessity. From $\sigma = 0$, it follows that $h_u(k_s)h_u = \frac{1}{n-1} \cdot \overline{g}[h_u(k_s)h_u] \cdot h_u = \frac{1}{n-1} \cdot \theta \cdot h_u$. Further, from the explicit form of h_u and k_s, it follows that

$$\begin{aligned} h_u(k_s)h_u &= \frac{1}{n-1} \cdot \theta \cdot h_u = g(k_s)g - \frac{1}{e} \cdot \{g(u) \otimes [g(u)](k_s)g + g(k_s)[g(u)] \otimes g(u)\} + \\ &\quad + \frac{1}{e^2} \cdot [g(u)](k_s)[g(u)] \cdot g(u) \otimes g(u) , \end{aligned}$$

or

$$\begin{aligned} k_s &= \frac{1}{e} \cdot \{u \otimes [g(u)](k_s) + (k_s)[g(u)] \otimes u\} - \frac{1}{e^2} \cdot [g(u)](k_s)[g(u)] \cdot u \otimes u + \\ &\quad + \frac{1}{n-1} \cdot \theta \cdot \overline{g}(h_u)\overline{g} . \end{aligned}$$

Since $[g(u)](k_s) = (k_s)[g(u)]$, $(k_s)[g(u)] = \frac{1}{2} \cdot \{(k)[g(u)] + [g(u)](k)\}$, $(k)[g(u)] = a$, $(k_s)[g(u)] = \frac{1}{2} \cdot \{a + [g(u)](k)\}$, $[g(u)](k_s)[g(u)] = [g(u)](k)[g(u)] =$

$g(u,a) = \frac{1}{2} \cdot [ue - (\nabla_u g)(u,u)]$, $\theta = \overline{g}[h_u(k_s)h_u] = \overline{g}[h_u(k)h_u]$, and $\overline{g}(h_u)\overline{g} = h^u$, the explicit form of k_s can be found as

$$\begin{aligned} k_s &= \frac{1}{2 \cdot e} \cdot \{u \otimes a + a \otimes u + u \otimes [g(u)](k) + [g(u)](k) \otimes u - \\ &\quad -\frac{1}{e} \cdot [ue - (\nabla_u g)(u,u)] \cdot u \otimes u\} + \\ &\quad +\frac{1}{n-1} \cdot \theta \cdot h^u \quad . \end{aligned}$$

2. Sufficiency. From the last expression and the above relations, it follows that $h_u(k_s)h_u = \frac{1}{n-1} \cdot \theta \cdot h_u$, and therefore $\sigma = 0$.

Proposition 6. A sufficient condition for the existence of a non-null vector field with vanishing shear velocity ($\sigma = 0$) and expansion velocity ($\theta = 0$) is the condition

$$h_u(k_s)h_u = 0 \ ,$$

identical with the condition

$$\begin{aligned} k_s &= \frac{1}{2 \cdot e} \cdot \{u \otimes a + a \otimes u + u \otimes [g(u)](k) + [g(u)](k) \otimes u - \\ &\quad -\frac{1}{e} \cdot [ue - (\nabla_u g)(u,u)] \cdot u \otimes u\} \end{aligned}$$

Proof. If $h_u(k_s)h_u = 0$, then $\theta = \overline{g}[h_u(k_s)h_u] = 0$. Therefore, $\sigma = h_u(k_s)h_u - \frac{1}{n-1} \cdot \theta \cdot h_u = 0$.

Corollary. If $h_u(k_s)h_u = 0$, then

$$g[k_s] = \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u,u)] \ .$$

Proof. It follows from the above proposition that

$$\begin{aligned} \theta &= g[k_s] - \frac{1}{e} \cdot g(u,a) = 0 \ , \\ g[k_s] &= \frac{1}{e} \cdot g(u,a) = \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u,u)] \ . \end{aligned}$$

2.3.3 Contravariant non-null vector fields with vanishing rotation velocity ($\omega = 0$)

If we consider the explicit form for ω

$$\omega = h_u(k_a)h_u$$

we can prove the following propositions:

Proposition 7. The necessary and sufficient condition for the existence of a contravariant non-null vector field with vanishing rotation velocity ($\omega = 0$) is the condition

$$k_a = \frac{1}{e} \cdot \{u \otimes [g(u)](k_a) - [g(u)](k_a) \otimes u\} \ , \tag{21}$$

or in a co-ordinate basis

$$k_a^{ij} = \frac{1}{e} \cdot g_{\overline{mn}} \cdot u^n \cdot (u^i \cdot k_a^{mj} - u^j \cdot k_a^{mi}) \quad . \tag{22}$$

Proof. 1. Necessity. Form $h_u(k_a)h_u = 0$, it follows that

$$\begin{aligned} h_u(k_a)h_u &= 0 = g(k_a)g - \frac{1}{e} \cdot g(u) \otimes [g(u)](k_a)g - \frac{1}{e} \cdot g(k_a)[g(u)] \otimes g(u) + \\ &\quad + \frac{1}{e^2} \cdot [g(u)](k_a)[g(u)] \cdot g(u) \cdot g(u) \ . \end{aligned}$$

Since

$$[g(u)](k_a)[g(u)] = g_{\overline{im}} \cdot u^m \cdot k_a^{ij} \cdot g_{\overline{jn}} \cdot u^n = -g_{\overline{im}} \cdot u^m \cdot k_a^{ij} \cdot g_{\overline{jn}} \cdot u^n \ ,$$

we have $[g(u)](k_a)[g(u)] = 0$. Therefore,

$$g(k_a)g = \frac{1}{e} \cdot \{g(u) \otimes [g(u)](k_a)g + g(k_a)[g(u)] \otimes g(u)\} \ .$$

From the last expression and from the relation $\overline{g}[g(k_a)g]\overline{g} = k_a$, it follows that

$$\begin{aligned} k_a &= \frac{1}{e} \cdot \{u \otimes [g(u)](k_a) + (k_a)[g(u)] \otimes u\} = \\ &= \frac{1}{e} \cdot \{u \otimes [g(u)](k_a) - [g(u)](k_a) \otimes u\} \ , \end{aligned}$$

because of $(k_a)[g(u)] = -\ [g(u)](k_a)$. In a co-ordinate basis we obtain (22).

2. Sufficiency. From ([**?**]) we have

$$g(k_a)g = \frac{1}{e} \cdot \{g(u) \otimes [g(u)](k_a)g + g(k_a)[g(u)] \otimes g(u)\} \ ,$$

which is identical to $h_u(k_a)h_u = 0$.

On the other hand, after direct computations, it follows that

$$(k_a)[g(u)] = \frac{1}{2} \cdot \{(k)[g(u)] - [g(u)](k)\} \ .$$

Since $(k)[g(u)] = a$, we have the relation

$$(k_a)[g(u)] = \frac{1}{2} \cdot \{a - [g(u)](k)\} \ .$$

Then

$$k_a = \frac{1}{2 \cdot e} \cdot \{a \otimes u - u \otimes a + u \otimes [g(u)](k) - [g(u)](k) \otimes u\} \ .$$

Proposition 8. A sufficient condition for the existence of a contravariant non-null vector field with vanishing rotation velocity ($\theta = 0$) is the condition

$$k_a = 0 \ .$$

Proof. If $k_a = 0$, then it follows directly from $\omega = h_u(k_a)h_u$ that $\omega = 0$.
In a co-ordinate basis k_a is equivalent to the expression

$$u^i{}_{;l} \cdot g^{lj} - u^j{}_{;l} \cdot g^{li} = (T_{lm}{}^i \cdot g^{lj} - T_{lm}{}^j \cdot g^{li}) \cdot u^m \ .$$

On the other side, after multiplying the last expression with $g_{\overline{jk}} \cdot u^k$ and summarizing over j, we obtain

$$a^i = u^i{}_{;k} \cdot u^k = g_{\overline{jk}} \cdot u^k \cdot (u^j{}_{;l} - T_{lm}{}^j \cdot u^m) \cdot g^{li} = g_{\overline{jk}} \cdot u^k \cdot k^{ji} \ ,$$

or in a form

$$a = [g(u)](k) \ .$$

Proposition 9. The necessary condition for $k_a = 0$ is the condition

$$a = [g(u)](k) \ .$$

Proof. From $k_a = 0$ and $(k_a)[g(u)] = \frac{1}{2} \cdot \{a - [g(u)](k)\}$, it follows that $a = [g(u)](k)$.

If the rotation velocity ω vanishes ($\omega = 0$) for an auto-parallel ($\nabla_u u = 0$) contravariant non-null vector field u, then the rotation acceleration tensor W will have the form [22]

$$W = \frac{1}{2} \cdot [h_u(\nabla_u \overline{g})\sigma - \sigma(\nabla_u \overline{g})h_u] \ .$$

From the last expression it is obvious that the nonmetricity ($\nabla_u g \neq 0$) in a $(\overline{L}_n, g)$-space is responsible for nonvanishing the rotation acceleration W.

2.4 Relative velocity and change of the length of a contravariant vector field over a (L_n, g)-space

Let us now consider the influence of the kinematic characteristics related to the relative velocity and respectively to the relative velocity, induced by the torsion, upon the change of the length of a contravariant vector field.

Let $l_\xi = \mid g(\xi, \xi) \mid^{\frac{1}{2}}$ be the length of a contravariant vector field ξ. The rate of change ul_ξ of l_ξ along a contravariant vector field u can be expressed in the form $\pm 2.l_\xi.(ul_\xi) = (\nabla_u g)(\xi, \xi) + 2g(\nabla_u \xi, \xi)$. By the use of the projections of ξ and $\nabla_u \xi$ along and orthogonal to u we can find the relations

$$2g(\nabla_u \xi, \xi) = 2.\tfrac{l}{e}.g(\nabla_u \xi, u) + 2g({}_{rel}v, \xi_\perp) \ ,$$
$$(\nabla_u g)(\xi, \xi) = (\nabla_u g)(\xi_\perp, \xi_\perp) + 2.\tfrac{l}{e}.(\nabla_u g)(\xi_\perp, u) + \tfrac{l^2}{e^2}.(\nabla_u g)(u, u) \ .$$

Then, it follows for $\pm 2.l_\xi.(ul_\xi)$ the expression

$$\begin{aligned} \pm 2.l_\xi.(ul_\xi) = (\nabla_u g)(\xi_\perp, \xi_\perp) + 2.\tfrac{l}{e}.[(\nabla_u g)(\xi_\perp, u) + g(\nabla_u \xi, u)] + \\ + \tfrac{l^2}{e^2}.(\nabla_u g)(u, u) + 2g({}_{rel}v, \xi_\perp) \ , \end{aligned} \tag{23}$$

where

$$g({}_{rel}v, \xi_\perp) = \frac{l}{e}.h_u(a, \xi_\perp) + h_u(\pounds_u \xi, \xi_\perp) + d(\xi_\perp, \xi_\perp) \ , \tag{24}$$

$$d(\xi_\perp, \xi_\perp) = \sigma(\xi_\perp, \xi_\perp) + \frac{1}{n-1}.\theta.l^2_{\xi_\perp} \ . \tag{25}$$

For finding out the last two expressions the following relations have been used:

$$g(\overline{g}(h_u)a, \xi_\perp) = h_u(a, \xi_\perp) \ , \quad g(\overline{g}(h_u)(\pounds_u \xi), \xi_\perp) = h_u(\pounds_u \xi, \xi_\perp) \ , \tag{26}$$

$$g(\overline{g}[d(\xi)], \xi_\perp) = d(\xi_\perp, \xi_\perp) \ , \quad d(\xi) = d(\xi_\perp) \ . \tag{27}$$

Special case: $g(u, \xi) = l := 0 : \xi = \xi_\perp$.

$$\pm\ 2.l_{\xi_\perp}.(ul_{\xi_\perp}) = (\nabla_u g)(\xi_\perp, \xi_\perp) + 2g({}_{rel}v, \xi_\perp) \ . \tag{28}$$

Special case: V_n-spaces: $\nabla_\eta g = 0$ for $\forall \eta \in T(M)$ $(g_{ij;k} = 0)$, $g(u, \xi) = l := 0 : \xi = \xi_\perp$.

$$\pm\ l_{\xi_\perp}.(ul_{\xi_\perp}) = g({}_{rel}v, \xi_\perp) \ . \tag{29}$$

In (L_n, g)-spaces the covariant derivative $\nabla_u g$ of the metric tensor field g along u can be decomposed in its trace free part ${}^s\nabla_u g$ and its trace part $\frac{1}{n}.Q_u.g$ as

$$\nabla_u g = {}^s\nabla_u g + \frac{1}{n}.Q_u.g \ , \qquad \dim M = n \ ,$$

where $\overline{g}[{}^s\nabla_u g] = 0, \quad Q_u = \overline{g}[\nabla_u g] = g^{kl}.g_{kl;j}.u^j = Q_j.u^j, \quad Q_j = g^{kl}.g_{kl;j}$.

Remark. The covariant vector $\overline{Q} = \frac{1}{n}.Q = \frac{1}{n}.Q_j.dx^j = \frac{1}{n}.Q_\alpha.e^\alpha$ is called *Weyl's covector field*. The operator ${}^s\nabla_u = \nabla_u - \frac{1}{n}.Q_u$ is called *trace free covariant operator*.

If we use now the decomposition of $\nabla_u g$ in the expression for $\pm 2.l_\xi.(ul_\xi)$ we find the relation

$$\begin{gathered}\pm 2.l_\xi.(ul_\xi) = ({}^s\nabla_u g)(\xi,\xi) + \tfrac{1}{n}.Q_u.l_\xi^2 + 2g(\nabla_u\xi,\xi) = \\ = ({}^s\nabla_u g)(\xi_\perp,\xi_\perp) + \\ + \tfrac{l}{e}.[2.({}^s\nabla_u g)(\xi_\perp,u) + 2.g(\nabla_u\xi,u) + \tfrac{l}{e}.({}^s\nabla_u g)(u,u)] + \\ + \tfrac{1}{n}.Q_u.(l_{\xi_\perp}^2 + \tfrac{l^2}{e}) + 2.g({}_{rel}v,\xi_\perp) \ ,\end{gathered} \qquad (30)$$

where $l_{\xi_\perp}^2 = g(\xi_\perp,\xi_\perp)$, $l = g(\xi,u)$.

For $l_\xi \neq 0$:

$$ul_\xi = \pm\frac{1}{2.l_\xi}({}^s\nabla_u g)(\xi,\xi) \pm \frac{1}{2.n}.Q_u.l_\xi \pm \frac{1}{l_\xi}.g(\nabla_u\xi,\xi) \ . \qquad (31)$$

In the case of a parallel transport $(\nabla_u\xi = 0)$ of ξ along u the change ul_ξ of the length l_ξ is

$$ul_\xi = \pm\frac{1}{2.l_\xi}({}^s\nabla_u g)(\xi,\xi) \pm \frac{1}{2.n}.Q_u.l_\xi \ . \qquad (32)$$

Special case: $\nabla_u\xi = 0$ and ${}^s\nabla_u g = 0$.

$$ul_\xi = \pm\frac{1}{2.n}.Q_u.l_\xi \ . \qquad (33)$$

Special case: $g(u,\xi) = l := 0 : \xi = \xi_\perp$.

$$\pm 2.l_{\xi_\perp}.(ul_{\xi_\perp}) = ({}^s\nabla_u g)(\xi_\perp,\xi_\perp) + \frac{1}{n}.Q_u.l_{\xi_\perp}^2 + 2.g({}_{rel}v,\xi_\perp) \ .$$

$$ul_{\xi_\perp} = \pm\frac{1}{2.l_{\xi_\perp}}.({}^s\nabla_u g)(\xi_\perp,\xi_\perp) \pm \frac{1}{2n}.Q_u.l_{\xi_\perp} \pm \frac{1}{l_{\xi_\perp}}.g({}_{rel}v,\xi_\perp) \ , \qquad l_{\xi_\perp} \neq 0 \ . \qquad (34)$$

Special case: Quasi-metric transports: $\nabla_u g := 2.g(u,\eta).g, \quad u,\eta \in T(M)$.

$$\pm 2.l_\xi.(ul_\xi) = 2.g(u,\eta).(l_{\xi_\perp}^2 + \frac{l^2}{e}) + 2.[\frac{l}{e}.g(\nabla_u\xi,u) + g({}_{rel}v,\xi_\perp)] \ . \qquad (35)$$

2.5 Change of the cosine between two contravariant vector fields and the relative velocity

The cosine between two contravariant vector fields ξ and η has been defined as $g(\xi,\eta) = l_\xi . l_\eta . \cos(\xi,\eta)$. The rate of change of the cosine along a contravariant vector field u can be found in the form

$$\begin{aligned} l_\xi . l_\eta . \{u[\cos(\xi,\eta)]\} = (\nabla_u g)(\xi,\eta) + g(\nabla_u \xi,\eta) + g(\xi,\nabla_u \eta) - \\ -[l_\eta .(u l_\xi) + l_\xi .(u l_\eta)] . \cos(\xi,\eta) \ . \end{aligned}$$

Special case: $\nabla_u \xi = 0$, $\nabla_u \eta = 0$, ${}^s\nabla_u g = 0$.

$$l_\xi . l_\eta . \{u[\cos(\xi,\eta)]\} = \frac{1}{n} . Q_u . g(\xi,\eta) - [l_\eta .(u l_\xi) + l_\xi .(u l_\eta)] . \cos(\xi,\eta) \ .$$

Since $g(\xi,\eta) = l_\xi . l_\eta . \cos(\xi,\eta)$, it follows from the last relation

$$l_\xi . l_\eta . \{u[\cos(\xi,\eta)]\} = \{\frac{1}{n} . Q_u . l_\xi . l_\eta - [l_\eta .(u l_\xi) + l_\xi .(u l_\eta)]\} . \cos(\xi,\eta) \ .$$

Therefore, if $\cos(\xi,\eta) = 0$ between two parallel transported along u vector fields ξ and η, then the right angle between them [determined by the condition $\cos(\xi,\eta) = 0$] does not change along the contravariant vector field u. In the cases, when $\cos(\xi,\eta) \neq 0$, the rate of change of the cosine of the angle between two vector fields ξ and η is linear to $\cos(\xi,\eta)$.

By the use of the definitions and the relations:

$$_{rel}v_\xi := \overline{g}[h_u(\nabla_u \xi)] = {}_{rel}v \ , \qquad {}_{rel}v_\eta := \overline{g}[h_u(\nabla_u \eta)] \ , \tag{36}$$

$$\begin{aligned} g(\nabla_u \xi,\eta) = \tfrac{1}{e} . g(u,\eta) . g(\nabla_u \xi, u) + g({}_{rel}v_\xi,\eta) \ , \\ g(\nabla_u \eta,\xi) = \tfrac{1}{e} . g(u,\xi) . g(\nabla_u \eta, u) + g({}_{rel}v_\eta,\xi) \ , \end{aligned} \tag{37}$$

$$(\nabla_u g)(\xi,\eta) = ({}^s\nabla_u g)(\xi,\eta) + \frac{1}{n} . Q_u . g(\xi,\eta) \ , \tag{38}$$

$$\begin{aligned} ({}^s\nabla_u g)(\xi,\eta) = ({}^s\nabla_u g)(\xi_\perp,\eta_\perp) + \tfrac{l}{e} . ({}^s\nabla_u g)(u,\eta_\perp) + \tfrac{\widetilde{l}}{e} . ({}^s\nabla_u g)(\xi_\perp,u) + \\ + \tfrac{l}{e} . \tfrac{\widetilde{l}}{e} . ({}^s\nabla_u g)(u,u) \ , \quad \widetilde{l} = g(u,\eta) \ , \quad \eta_\perp = \overline{g}[h_u(\eta)] \ , \quad l = g(u,\xi) \ , \end{aligned} \tag{39}$$

$$\begin{aligned} (\nabla_u g)(\xi,\eta) = ({}^s\nabla_u g)(\xi,\eta) + \tfrac{1}{n} . Q_u . g(\xi,\eta) = \\ = ({}^s\nabla_u g)(\xi_\perp,\eta_\perp) + \tfrac{l}{e} . ({}^s\nabla_u g)(u,\eta_\perp) + \tfrac{\widetilde{l}}{e} . ({}^s\nabla_u g)(\xi_\perp,u) + \\ + \tfrac{l}{e} . \tfrac{\widetilde{l}}{e} . ({}^s\nabla_u g)(u,u) + \tfrac{1}{n} . Q_u . [\tfrac{l . \widetilde{l}}{e} + g(\xi_\perp,\eta_\perp)] \ , \end{aligned} \tag{40}$$

the expression of $l_\xi.l_\eta.\{u[\cos(\xi,\eta)]\}$ follows in the form

$$\begin{array}{c} l_\xi.l_\eta.\{u[\cos(\xi,\eta)]\} = ({}^s\nabla_u g)(\xi_\perp,\eta_\perp) + \frac{l}{e}.[({}^s\nabla_u g)(u,\eta_\perp) + g(\nabla_u\eta,u)] + \\ +\frac{\widetilde{l}}{e}.[({}^s\nabla_u g)(\xi_\perp,u) + g(\nabla_u\xi,u)] + \frac{l.\widetilde{l}}{e^2}.({}^s\nabla_u g)(u,u) + \\ +\frac{1}{n}.Q_u.[\frac{l.\widetilde{l}}{e} + g(\xi_\perp,\eta_\perp)] + g({}_{rel}v_\xi,\eta) + g({}_{rel}v_\eta,\xi) - \\ -[l_\eta.(ul_\xi) + l_\xi.(ul_\eta)].\cos(\xi,\eta)\ . \end{array} \tag{41}$$

Special case: $g(u,\xi) = l := 0$, $g(u,\eta) = \widetilde{l} := 0 : \xi = \xi_\perp$, $\eta = \eta_\perp$.

$$\begin{array}{c} l_{\xi_\perp}.l_{\eta_\perp}.\{u[\cos(\xi_\perp,\eta_\perp)]\} = ({}^s\nabla_u g)(\xi_\perp,\eta_\perp) + \frac{1}{n}.Q_u.l_{\xi_\perp}.l_{\eta_\perp}.\cos(\xi_\perp,\eta_\perp) + \\ + g({}_{rel}v_{\xi_\perp},\eta_\perp) + g({}_{rel}v_{\eta_\perp},\xi_\perp) - [l_{\eta_\perp}.(ul_{\xi_\perp}) + l_{\xi_\perp}.(ul_{\eta_\perp})].\cos(\xi_\perp,\eta_\perp)\ , \end{array} \tag{42}$$

where $g(\xi_\perp,\eta_\perp) = l_{\xi_\perp}.l_{\eta_\perp}.\cos(\xi_\perp,\eta_\perp)$.

Special case: ${}^s\nabla_u g := 0 : \nabla_u g = \frac{1}{n}.Q_u.g$ (Weyl's space with torsion).

$$\pm 2.l_\xi.(ul_\xi) = 2.\frac{l}{e}.g(\nabla_u\xi,u) + \frac{1}{n}.Q_u.(l^2_{\xi_\perp} + \frac{l^2}{e}) + 2.g({}_{rel}v,\xi_\perp)\ , \tag{43}$$

$$g(\nabla_u\xi,u) = ul - \frac{1}{n}.Q_u.l - g(\xi,a)\ , \qquad\qquad a = \nabla_u u\ , \tag{44}$$

$$\begin{array}{c} l_\xi.l_\eta.\{u[\cos(\xi,\eta)]\} = \frac{l}{e}.g(\nabla_u\eta,u) + \frac{\widetilde{l}}{e}.g(\nabla_u\xi,u) + \\ +\frac{1}{n}.Q_u.[\frac{l.\widetilde{l}}{e} + g(\xi_\perp,\eta_\perp)] + g({}_{rel}v_\xi,\eta) + g({}_{rel}v_\eta,\xi) - \\ -[l_\eta.(ul_\xi) + l_\xi.(ul_\eta)].\cos(\xi,\eta)\ , \end{array} \tag{45}$$

Special case: ${}^s\nabla_u g := 0 : \nabla_u g = \frac{1}{n}.Q_u.g$, $g(u,\xi) = l := 0$, $g(u,\eta) = \widetilde{l} := 0$ (Weyl's space with torsion and orthogonal to u vector fields $\xi_\perp$ and $\eta_\perp$).

$$\pm 2.l_{\xi_\perp}.(ul_{\xi_\perp}) = \frac{1}{n}.Q_u.l^2_{\xi_\perp} + 2.g({}_{rel}v,\xi_\perp)\ , \tag{46}$$

$$\begin{array}{c} l_{\xi_\perp}.l_{\eta_\perp}.\{u[\cos(\xi_\perp,\eta_\perp)]\} = \frac{1}{n}.Q_u.g(\xi_\perp,\eta_\perp) + \\ + g({}_{rel}v_{\xi_\perp},\eta_\perp) + g({}_{rel}v_{\eta_\perp},\xi_\perp) - [l_{\eta_\perp}.(ul_{\xi_\perp}) + l_{\xi_\perp}.(ul_{\eta_\perp})].\cos(\xi_\perp,\eta_\perp) = \\ = g({}_{rel}v_{\xi_\perp},\eta_\perp) + g({}_{rel}v_{\eta_\perp},\xi_\perp) - \\ - [l_{\eta_\perp}.(ul_{\xi_\perp}) + l_{\xi_\perp}.(ul_{\eta_\perp}) + \frac{1}{n}.Q_u.l_{\xi_\perp}.l_{\eta_\perp}].\cos(\xi_\perp,\eta_\perp) \end{array} \tag{47}$$

Special case: U_n-spaces: $\nabla_\xi g = 0$ for $\forall\xi \in T^*(M)$:

$$\begin{array}{c} l_{\xi_\perp}.l_{\eta_\perp}.\{u[\cos(\xi_\perp,\eta_\perp)]\} = \\ = g({}_{rel}v_{\xi_\perp},\eta_\perp) + g({}_{rel}v_{\eta_\perp},\xi_\perp) - [l_{\eta_\perp}.(ul_{\xi_\perp}) + l_{\xi_\perp}.(ul_{\eta_\perp})].\cos(\xi_\perp,\eta_\perp). \end{array}$$

2.6 Rate of change of the length of a vector field $\xi_\perp$ connecting two particles (points) in the space-time as a U_n-space

The distance between a particle (as a basic point) and an other particle (as an observed point) lying in a neighborhood of the first one can be determined by the use of the length of the vector field $\xi_\perp$. The rate of change of the length $l_{\xi_\perp}$ of the vector field $\xi_\perp$ (along the vector field u) in a U_n-space can be given in the form

$$ul_{\xi_\perp} = \pm \frac{1}{l_{\xi_\perp}}.g(\nabla_u \xi_\perp, \xi_\perp) \ , \quad l_{\xi_\perp} \neq 0 \ .$$

If u is tangential vector field along a congruence of curves with parameter s, i. e. if $u = \frac{d}{ds}$, then $ul_{\xi_\perp}$ will have the form

$$\frac{dl_{\xi_\perp}}{ds} = \pm \frac{1}{l_{\xi_\perp}}.g(\nabla_u \xi_\perp, \xi_\perp) = \pm \frac{1}{l_{\xi_\perp}}.g_{ij}.\xi^i_{\perp\ ;k}.u^k.\xi^j_\perp \ .$$

By the use of the decomposition of $\nabla_u \xi_\perp$,

$$\nabla_u \xi_\perp = \frac{\bar{l}}{e}.u + {}_{rel}v \quad , \quad \bar{l} = g(\nabla_u \xi_\perp, u) \ ,$$

under the condition for the orthogonality between u and $\xi_\perp : g(u, \xi_\perp) = l = 0$, we obtain

$$\frac{dl_{\xi_\perp}}{ds} = \pm \frac{1}{l_{\xi_\perp}}.g({}_{rel}v, \xi_\perp) \ .$$

Under the assumption $\pounds_{\xi_\perp} u = - \pounds_u \xi_\perp = 0$, it follows that

$$\frac{dl_{\xi_\perp}}{ds} = \pm \frac{1}{l_{\xi_\perp}}.g({}_{rel}v, \xi_\perp) = \pm \frac{1}{l_{\xi_\perp}}.d(\xi_\perp, \xi_\perp) \ . \tag{48}$$

If we use the explicit form for d and the fact that $\omega(\xi_\perp, \xi_\perp) = 0$, then we can find the expression for the rate of change of $l_{\xi_\perp}$ in the form

$$\frac{dl_{\xi_\perp}}{ds} = \pm \frac{1}{l_{\xi_\perp}}.d(\xi_\perp, \xi_\perp) = \pm \frac{1}{l_{\xi_\perp}}.\sigma(\xi_\perp, \xi_\perp) \pm \frac{1}{n-1}.\theta.l_{\xi_\perp} \ . \tag{49}$$

Remark. The sign $\pm$ depends on the sign of the metric g (for $n = 4$, sign $g = \pm 2$).

Since the existence of the torsion could cause the condition $d = 0$, respectively $\sigma = 0$, and $\theta = 0$, the length $l_{\xi_\perp}$ would not change along u and therefore, along the proper time $\tau = \frac{s}{c}$, if we consider τ as the proper time of the basic particle (point).

3 Relative acceleration and its kinematic characteristics induced by the torsion

3.1 Relative acceleration in (L_n, g)-spaces

The notion *relative acceleration* vector field (relative acceleration) ${}_{rel}a$ in (L_n, g)-spaces can be defined (in analogous way as ${}_{rel}v$) (regardless of its physical interpretation) as the orthogonal to a non-null contravariant vector field u $[g(u,u) = e \neq 0]$ projection of the second covariant derivative (along the same non-null vector field u) of (another) contravariant vector field ξ, i.e.

$$ {}_{rel}a = \overline{g}(h_u(\nabla_u \nabla_u \xi)) = g^{ij}.h_{jk}.(\xi^k{}_{;l}.u^l)_{;m}.u^m.e_i \ . \tag{50} $$

$\nabla_u \nabla_u \xi = (\xi^i{}_{;l}.u^l)_{;m}.u^m.e_i$ is the second covariant derivative of a vector field ξ along the vector field u. It is an essential part of all types of deviation equations in V_n- and (L_n, g)-spaces [29] , [30].

If we take into account the expression for $\nabla_u \xi : \nabla_u \xi = k[g(\xi)] - \pounds_\xi u$, and differentiate covariant along u, then we obtain

$$ \nabla_u \nabla_u \xi = \{\nabla_u[(k)g]\}(\xi) + (k)(g)(\nabla_u \xi) - \nabla_u(\pounds_\xi u) \ . $$

By means of the relations

$$ k(g)\overline{g} = k \ , \qquad \nabla_u[k(g)] = (\nabla_u k)(g) + k(\nabla_u g) \ , $$
$$ \{\nabla_u[k(g)]\}\overline{g} = \nabla_u k + k(\nabla_u g)\overline{g} \ , $$

$\nabla_u \nabla_u \xi$ can be written in the form

$$ \nabla_u \nabla_u \xi = \frac{l}{e}.H(u) + B(h_u)\xi - k(g)\pounds_\xi u - \nabla_u(\pounds_\xi u) \tag{51} $$

[compare with $\nabla_u \xi = \frac{l}{e}.a + k(h_u)\xi - \pounds_\xi u$],
where

$$ H = B(g) = (\nabla_u k)(g) + k(\nabla_u g) + k(g)k(g) \ , $$

$$B = \nabla_u k + k(g)k + k(\nabla_u g)\overline{g} = \nabla_u k + k(g)k - k(g)(\nabla_u \overline{g}) \; .$$

The orthogonal to u covariant projection of $\nabla_u \nabla_u \xi$ will have therefore the form

$$h_u(\nabla_u \nabla_u \xi) = h_u[\frac{l}{e} H(u) - k(g)\pounds_\xi u - \nabla_u \pounds_\xi u] + [h_u(B)h_u](\xi) \; . \tag{52}$$

In the special case, when $g(u, \xi) = l = 0$ and $\pounds_\xi u = 0$, the above expression has the simple form

$$g({}_{rel}a) = h_u(\nabla_u \nabla_u \xi) = [h_u(B)h_u](\xi) = A(\xi) \; , \tag{53}$$

[compare with $h_u(\nabla_u \xi) = [h_u(k)h_u](\xi) = d(\xi)$].

The explicit form of $H(u)$ follows from the explicit form of H and its action on the vector field u

$$H(u) = (\nabla_u k)[g(u)] + k(\nabla_u g)(u) + k(g)(a) = \nabla_u[k(g)(u)] = \nabla_u a \; . \tag{54}$$

Now $h_u[\nabla_u \nabla_u \xi]$ can be written in the form

$$h_u(\nabla_u \nabla_u \xi) = h_u[\frac{l}{e} . \nabla_u a - k(g)(\pounds_\xi u) - \nabla_u(\pounds_\xi u)] + A(\xi) \tag{55}$$

[compare $h_u(\nabla_u \xi) = h_u(\frac{l}{e}.a - \pounds_\xi u) + d(\xi)$].

The explicit form of $A = h_u(B)h_u$ can be found in an analogous way as the explicit form for $d = h_u(k)h_u$ in the expression for ${}_{rel}v$.

3.2 Deformation acceleration, shear acceleration, rotation acceleration and expansion acceleration

The covariant tensor A, named *deformation acceleration* tensor can be represented as a sum, containing three terms: a trace-free symmetric term, an antisymmetric term and a trace term

$$A = {}_sD + W + \frac{1}{n-1}.U.h_u \tag{56}$$

where

$$D = h_u({}_sB)h_u \; , \qquad W = h_u({}_aB)h_u \; , \qquad U = \overline{g}[{}_sA] = \overline{g}[D] \tag{57}$$

$$ {}_sB = \frac{1}{2}(B^{ij} + B^{ji}).e_i.e_j \ , \qquad {}_aB = \frac{1}{2}(B^{ij} - B^{ji}).e_i \wedge e_j, \tag{58} $$

$$ {}_sD = D - \frac{1}{n-1}.\overline{g}[D].h_u = D - \frac{1}{n-1}.U.h_u \ . \tag{59} $$

The tensor ${}_sD$ is the *shear acceleration* tensor (shear acceleration), W is the *rotation acceleration* tensor (rotation acceleration) and U is the *expansion acceleration* invariant (expansion acceleration). Furthermore, every one of these quantities can be divided into three parts: torsion- and curvature-free acceleration, acceleration induced by the torsion and acceleration induced by the curvature.

Let us now consider the representation of every acceleration quantity in its essential parts connected with its physical interpretation.

The deformation acceleration tensor A can be written in the following forms

$$ A = {}_sD + W + \frac{1}{n-1}.U.h_u = A_0 + G = {}_FA_0 - {}_TA_0 + G \ , \tag{60} $$

$$ A = {}_sD_0 + W_0 + \frac{1}{n-1}.U_0.h_u + {}_sM + N + \frac{1}{n-1}.I.h_u \ , \tag{61} $$

$$ \begin{aligned} A = {}_{sF}D_0 + {}_FW_0 + \tfrac{1}{n-1}.{}_FU_0.h_u - ({}_{sT}D_0 + {}_TW_0 + \tfrac{1}{n-1}.{}_TU_0.h_u) + \\ + {}_sM + N + \tfrac{1}{n-1}.I.h_u \ , \end{aligned} \tag{62} $$

where A_0 is curvature-free deformation acceleration tensor, G is the deformation acceleration tensor induced by the curvature, ${}_FA_0$ is the torsion-free and curvature-free deformation acceleration tensor, ${}_TA_0$ is the deformation acceleration tensor induced by the torsion. Every of these tensors could be represented in its three parts (the symmetric trace-free part, the antisymmetric part and the trace part) determining the corresponding shear acceleration, rotation acceleration and expansion acceleration:

$$ A_0 = {}_FA_0 - {}_TA_0 = {}_sD_0 + W_0 + \frac{1}{n-1}.U_0.h_u \ , \tag{63} $$

$$ {}_FA_0 = {}_{sF}D_0 + {}_FW_0 + \frac{1}{n-1}.{}_FU_0.h_u \ , \qquad {}_FA_0(\xi) = h_u(\nabla_{\xi_\perp} a) \ , \tag{64} $$

$$ {}_TA_0 = {}_{sT}D_0 + {}_TW_0 + \frac{1}{n-1}.{}_TU_0.h_u \ , \tag{65} $$

$$ G = {}_sM + N + \frac{1}{n-1}.I.h_u = h_u(K)h_u \ , \tag{66} $$

$$h_u([R(u,\xi)]u) = h_u(K)h_u(\xi) \text{ for } \forall\ \xi \in T(M)\ , \tag{67}$$

$$[R(u,\xi)]u = \nabla_u\nabla_\xi u - \nabla_\xi\nabla_u u - \nabla_{\pounds_u\xi} u\ , \tag{68}$$

$$K = K^{kl}.e_k \otimes e_l\ ,\ K^{kl} = R^k{}_{mnr}.g^{rl}.u^m.u^n\ , \tag{69}$$

The components $R^k{}_{mnr}$ are the components of the contravariant Riemannian curvature tensor,

$$K_a = K_a^{kl}.e_k \wedge e_l,\ K_a^{kl} = \frac{1}{2}(K^{kl} - K^{lk})\ ,\ K_s = K_s^{kl}.e_k.e_l\ ,\ K_s^{kl} = \frac{1}{2}(K^{kl} + K^{lk})\ , \tag{70}$$

$${}_sD = {}_sD_0 + {}_sM\ ,\ W = W_0 + N = {}_FW_0 - {}_TW_0 + N\ , \tag{71}$$

$$U = U_0 + I = {}_FU_0 - {}_TU_0 + I\ , \tag{72}$$

$${}_sM = M - \frac{1}{n-1}.I.h_u\ ,\ M = h_u(K_s)h_u\ ,\ I = \overline{g}[M] = g^{\overline{ij}}.M_{ij}\ , \tag{73}$$

$$N = h_u(K_a)h_u\ , \tag{74}$$

$${}_sD_0 = {}_{sF}D_0 - {}_{sT}D_0 = {}_FD_0 - \frac{1}{n-1}.{}_FU_0.h_u - ({}_TD_0 - \frac{1}{n-1}.{}_TU_0.h_u)\ , \tag{75}$$

$${}_sD_0 = {}_{sF}D_0 - {}_TD_0 - \frac{1}{n-1}({}_FU_0 -_T U_0)h_u\ , \tag{76}$$

$${}_sD_0 = D_0 - \frac{1}{n-1}.U_0.h_u\ , \tag{77}$$

$${}_{sF}D_0 = {}_FD_0 - \frac{1}{n-1}.{}_FU_0.h_u\ ,\ {}_FD_0 = h_u(b_s)h_u\ , \tag{78}$$

$$b = b_s + b_a\ , \quad b = b^{kl}.e_k \otimes e_l\ , \quad b^{kl} = a^k{}_{;n}.g^{nl}\ , \tag{79}$$

$$a^k = u^k{}_{;m}.u^m\ , \quad b_s = b_s^{kl}.e_k.e_l\ , \quad b_s^{kl} = \frac{1}{2}(b^{kl} + b^{lk})\ , \tag{80}$$

$$b_a = b_a^{kl}.e_k \wedge e_l\ , \quad b_a^{kl} = \frac{1}{2}(b^{kl} - b^{lk})\ , \tag{81}$$

$${}_FU_0 = \overline{g}[{}_FD_0] = g[b] - \frac{1}{e}.g(u, \nabla_u a)\ ,\ g[b] = g_{kl}.b^{kl}\ , \tag{82}$$

$${}_{sT}D_0 = {}_TD_0 - \frac{1}{n-1}.{}_TU_0.h_u = {}_{sF}D_0 - {}_sD_0\ ,\ {}_TD_0 = {}_FD_0 - D_0\ , \tag{83}$$

$$U_0 = \overline{g}[D_0] = {}_FU_0 - {}_TU_0\ , \quad {}_TU_0 = \overline{g}[{}_TD_0]\ , \tag{84}$$

$${}_FW_0 = h_u(b_a)h_u\ , \quad {}_TW_0 = {}_FW_0 - W_0\ . \tag{85}$$

Under the conditions $\pounds_\xi u = 0$, $\xi = \xi_\perp = \overline{g}(h_u(\xi))$, $(l = 0)$, the expression for $h_u(\nabla_u \nabla_v \xi)$ can be written in the forms

$$g({}_{rel}a) = h_u(\nabla_u \nabla_u \xi_\perp) = A(\xi_\perp) = A_0(\xi_\perp) + G(\xi_\perp) \ , \tag{86}$$

$$g({}_{rel}a) = h_u(\nabla_u \nabla_u \xi_\perp) = {}_FA_0(\xi_\perp) - {}_TA_0(\xi_\perp) + G(\xi_\perp) \ , \tag{87}$$

$$g({}_{rel}a) = h_u(\nabla_u \nabla_u \xi_\perp) = ({}_{sF}D_0 + {}_FW_0 + \frac{1}{n-1}.{}_FU_0.g)(\xi_\perp) -$$

$$- ({}_{sT}D_0 + {}_TW_0 + \frac{1}{n-1}.{}_TU_0.g)(\xi_\perp) + ({}_sM + N + \frac{1}{n-1}.I.g)(\xi_\perp) \ , \tag{88}$$

which enable one to find a physical interpretation of the quantities ${}_sD$,W,U and of the quantities${}_{sF}D_0$, ${}_FW_0$, ${}_FU_0$, ${}_{sT}D_0$, ${}_TW_0$, ${}_TU_0$, ${}_sM$, N, I contained in their structure [22].

After the above consideration the following proposition can be formulated:

Proposition 10. The covariant vector field $g({}_{rel}a) = h_u(\nabla_u \nabla_u \xi)$ can be written in the form

$$g({}_{rel}a) = h_u[\frac{l}{e}.\nabla_u a - \nabla_{\pounds_\xi u} u - \nabla_u(\pounds_\xi u) + T(\pounds_\xi u, u)] + A(\xi) \ ,$$

with $A(\xi) = {}_sD(\xi) + W(\xi) + \frac{1}{n-1}.U.h_u(\xi)$.

For the case of affine symmetric (Levi-Civita) connection $[T(w, v) = 0$ for $\forall\ w, v \in T(M)$, $T_{ij}{}^k = 0$, $\Gamma^k_{ij} = \Gamma^k_{ji}]$ and Riemannian metric ($\nabla_v g = 0$ for $\forall v \in T(M)$, $g_{ij;k} = 0$), kinematic characteristics are obtained in V_n-spaces connected with the notion relative velocity [21] , and the relative acceleration [31] [32] . For the case of affine non-symmetric connection $[T(w, v) \neq 0$ for $\forall$ $w, v \in T(M)$, $\Gamma^i_{jk} \neq \Gamma^i_{kj}]$ and Riemannian metric kinematic characteristics are obtained in U_n-spaces [31].

The shear, rotation and expansion accelerations induced by the torsion can be expressed by means of the kinematic characteristics of the relative velocity [22]:

(a) Shear acceleration tensor induced by the torsion ${}_{sT}D_0$

$${}_{sT}D_0 = {}_TD_0 - \frac{1}{n-1}.{}_TU_0.h_u$$

$$
\begin{array}{c}
{}_T D_0 = \frac{1}{2}[{}_sP(\overline{g})\sigma + \sigma(\overline{g})_sP] + \frac{1}{2}[Q(\overline{g})\omega + \omega(\overline{g})Q] + \\
+\frac{1}{n-1}(\theta_1.\sigma + \theta._sP) + \frac{1}{n-1}(\dot{\theta_1} + \frac{1}{n-1}.\theta_1.\theta)h_u + \nabla_u({}_sP) + \\
+\frac{1}{2}[{}_sP(\overline{g})\omega - \omega(\overline{g})_sP] + \frac{1}{2}[Q(\overline{g})\sigma - \sigma(\overline{g})Q] + \\
+\frac{1}{2e}[h_u(a) \otimes (g(u))(m+q)h_u + h_u((g(u))(m+q)) \otimes h_u(a)] + \\
+\frac{1}{e}[{}_sP(a) \otimes g(u) + g(u) \otimes_s P(a)] + \\
+\frac{1}{2}[h_u(\nabla_u\overline{g})_sP + {}_sP(\nabla_u\overline{g})h_u] + \frac{1}{2}[h_u(\nabla_u g)Q - Q(\nabla_u g)h_u] \ .
\end{array}
\tag{89}
$$

(b) Expansion acceleration induced by the torsion ${}_TU_0$ ($\dot{\theta_1} := u\theta_1$)

$$
{}_TU_0 = \overline{g}[{}_sP(\overline{g})\sigma] + \overline{g}[Q(\overline{g})\omega] + \dot{\theta_1} + \frac{1}{n-1}.\theta_1.\theta + \frac{1}{e}.g(u, T(a,u)) \ . \tag{90}
$$

(c) Rotation acceleration tensor induced by the torsion ${}_TW_0$

$$
\begin{array}{c}
{}_TW_0 = \frac{1}{2}[{}_sP(\overline{g})\sigma - \sigma(\overline{g})_sP] + \frac{1}{2}[Q(\overline{g})\omega - \omega(\overline{g})Q)] + \\
+\frac{1}{n-1}(\theta_1.\omega + \theta.Q) + \nabla_uQ + \frac{1}{2}[{}_sP(\overline{g})\omega + \omega(\overline{g})_sP] + \\
+\frac{1}{2}[Q(\overline{g})\sigma + \sigma(\overline{g})Q] + \\
+\frac{1}{2e}[h_u(a) \otimes (g(u))(m+q)h_u - h_u((g(u))(m+q)) \otimes h_u(a)] + \\
+\frac{1}{e}[Q(a) \otimes g(u) - g(u) \otimes Q(a)] + \\
+\frac{1}{2}[h_u(\nabla_u\overline{g})_sP - {}_sP(\nabla_u\overline{g})h_u] + \frac{1}{2}[h_u(\nabla_u\overline{g})Q + Q(\nabla_u\overline{g})h_u] \ .
\end{array}
\tag{91}
$$

The kinematic characteristics related to the notion of relative acceleration can be used in finding out their influence on the rate of change of the length of a contravariant vector field.

3.3 Relative acceleration and change of the change of the length of a contravariant vector field over a (L_n, g)-space

The rate of change ul_ξ of the length $l_\xi = \mid g(\xi,\xi) \mid^{\frac{1}{2}}$ of a contravariant vector field ξ along a contravariant vector field u can be written in the form $\pm 2.l_\xi.(ul_\xi) = (\nabla_u g)(\xi,\xi) + 2g(\nabla_u\xi,\xi)$. After a further differentiation of the last expression along the vector field u we find the relation

$$
\begin{array}{c}
\pm(ul_\xi)^2 \pm l_\xi.(u(ul_\xi)) = \frac{1}{2}.(\nabla_u\nabla_u g)(\xi,\xi) + g(\nabla_u\nabla_u\xi,\xi) + \\
+2.(\nabla_u g)(\nabla_u\xi,\xi) + g(\nabla_u\xi,\nabla_u\xi) \ .
\end{array}
\tag{92}
$$

Our aim is to represent the terms of the right side of the last expression by the use of the projective metrics of the vector field u and the decomposition

of the metric tensor g in its trace free and trace parts. Since we already have the representation of this type for ul_ξ (see the above section), we will find the corresponding representation for $u(ul_\xi) = uul_\xi$. After some computations, the following relations can be found:

(a) Representation of $(\nabla_u \nabla_u g)(\xi, \xi)$. By the use of the expressions:

$$ {}^s\nabla_u Q_u = uQ_u - \frac{1}{n}.Q_u^2 \ , \tag{93} $$

$$ \nabla_u \nabla_u g = {}^s\nabla_u {}^s\nabla_u g + \frac{1}{n}.[(uQ_u).g + 2.Q_u.({}^s\nabla_u g)] \ , \tag{94} $$

$$ (\nabla_u \nabla_u g)(\xi, \xi) = (\nabla_u \nabla_u g)(\xi_\perp, \xi_\perp) + \frac{l^2}{e^2}.(\nabla_u \nabla_u g)(u, u) + 2.\frac{l}{e}.(\nabla_u \nabla_u g)(u, \xi_\perp) \ , \tag{95} $$

$$ (\nabla_u \nabla_u g)(\xi_\perp, \xi_\perp) = ({}^s\nabla_u {}^s\nabla_u g)(\xi_\perp, \xi_\perp) + \frac{2}{n}.Q_u.({}^s\nabla_u g)(\xi_\perp, \xi_\perp) + \frac{1}{n}.(uQ_u).l^2_{\xi_\perp} \ , \tag{96} $$

$$ \frac{l^2}{e^2}.(\nabla_u \nabla_u g)(u, u) = \frac{l^2}{e^2}.({}^s\nabla_u {}^s\nabla_u g)(u, u) + \frac{2}{n}.\frac{l^2}{e^2}.Q_u.({}^s\nabla_u g)(u, u) + \frac{1}{n}.\frac{l^2}{e}.(uQ_u) \ , \tag{97} $$

$$ (\nabla_u \nabla_u g)(u, \xi_\perp) = ({}^s\nabla_u {}^s\nabla_u g)(u, \xi_\perp) + \frac{2}{n}.Q_u.({}^s\nabla_u g)(u, \xi_\perp) \ , \tag{98} $$

the expression for $(\nabla_u \nabla_u g)(\xi, \xi)$ follows in the form

$$ \begin{gathered} (\nabla_u \nabla_u g)(\xi, \xi) = ({}^s\nabla_u {}^s\nabla_u g)(\xi_\perp, \xi_\perp) + 2.\tfrac{l}{e}.({}^s\nabla_u {}^s\nabla_u g)(u, \xi_\perp) + \\ + \tfrac{l^2}{e^2}.({}^s\nabla_u {}^s\nabla_u g)(u, u) + \tfrac{2}{n}.Q_u.[({}^s\nabla_u g)(\xi_\perp, \xi_\perp) + 2.\tfrac{l}{e}.({}^s\nabla_u g)(u, \xi_\perp)] + \\ + \tfrac{1}{n}.(uQ_u).(l^2_{\xi_\perp} + \tfrac{l^2}{e}) + \tfrac{2}{n}.\tfrac{l^2}{e^2}.Q_u.({}^s\nabla_u g)(u, u) \ . \end{gathered} \tag{99} $$

(b) Representation of $g(\nabla_u \nabla_u \xi, \xi)$:

$$ g(\nabla_u \nabla_u \xi, \xi) = g({}_{rel}a, \xi_\perp) + \frac{l}{e}.g(u, \nabla_u \nabla_u \xi) \ , \tag{100} $$

(c) Representation of $(\nabla_u g)(\nabla_u \xi, \xi)$. By the use of the expressions

$$ \begin{gathered} (\nabla_u g)(\nabla_u \xi, \xi) = \tfrac{l}{e^2}.(\nabla_u g)(u, u).g(\nabla_u \xi, u) + \tfrac{1}{e}.(\nabla_u g)(u, \xi_\perp).g(\nabla_u \xi, u) + \\ + \tfrac{l}{e}.(\nabla_u g)({}_{rel}v, u) + (\nabla_u g)({}_{rel}v, \xi_\perp) \ , \qquad {}_{rel}v = \overline{g}[h_u(\nabla_u \xi)] \ , \end{gathered} \tag{101} $$

and the representation of $\nabla_u g$ by the use of ${}^s\nabla_u g$ and Q_u, we obtain for $(\nabla_u g)(\nabla_u \xi, \xi)$

$$\begin{gathered}(\nabla_u g)(\nabla_u \xi, \xi) = \frac{1}{e}.({}^s\nabla_u g)(u, \xi_\perp).g(\nabla_u \xi, u) + \frac{l}{e^2}.({}^s\nabla_u g)(u, u).g(\nabla_u \xi, u) + \\ + ({}^s\nabla_u g)({}_{rel}v, \xi_\perp) + \frac{l}{e}.({}^s\nabla_u g)({}_{rel}v, u) + \frac{1}{n}.Q_u.[g({}_{rel}v, \xi_\perp) + \frac{l}{e}.g(\nabla_u \xi, u)] \ .\end{gathered} \tag{102}$$

(d) Representation of $g(\nabla_u \xi, \nabla_u \xi)$:

$$g(\nabla_u \xi, \nabla_u \xi) = g({}_{rel}v, {}_{rel}v) + \frac{1}{e}.[g(\nabla_u \xi, u)]^2 \ , \tag{103}$$

where

$$g(\nabla_u \xi, \nabla_u \xi) = l^2_{\nabla_u \xi} \ , \qquad g({}_{rel}v, {}_{rel}v) = l^2_{{}_{rel}v} \ , \qquad e = g(u, u) = l_u^2 \neq 0 \ , \tag{104}$$

$$g(\nabla_u \xi, u) = ul - \frac{1}{n}.Q_u.l - ({}^s\nabla_u g)(\xi, u) - g(\xi, a) \ , \qquad a = \nabla_u u \ . \tag{105}$$

By the use of all main results from (a) - (d) we find the explicit form for $u(ul_\xi)$

$$\begin{gathered}\pm\, l_\xi.(u(ul_\xi)) = \mp(ul_\xi)^2 + \frac{1}{2}.[({}^s\nabla_u {}^s\nabla_u g)(\xi_\perp, \xi_\perp) + \frac{l^2}{e^2}.({}^s\nabla_u {}^s\nabla_u g)(u, u)] + \\ + \frac{l}{e}.({}^s\nabla_u {}^s\nabla_u g)(u, \xi_\perp) + \frac{1}{n}.Q_u.[({}^s\nabla_u g)(\xi_\perp, \xi_\perp) + 2.\frac{l}{e}.({}^s\nabla_u g)(u, \xi_\perp) + \\ + \frac{l^2}{e^2}.({}^s\nabla_u g)(u, u) + 2.g({}_{rel}v, \xi_\perp) + 2.\frac{l}{e}.g(\nabla_u \xi, u)] + \frac{1}{2n}.(uQ_u).(l^2_{\xi_\perp} + \frac{l^2}{e}) + \\ + \frac{2}{e}.({}^s\nabla_u g)(u, \xi_\perp).g(\nabla_u \xi, u) + 2.\frac{l}{e^2}.({}^s\nabla_u g)(u, u).g(\nabla_u \xi, u) + \\ +2.({}^s\nabla_u g)({}_{rel}v, \xi_\perp) + 2.\frac{l}{e}.({}^s\nabla_u g)({}_{rel}v, u) + g({}_{rel}v, {}_{rel}v) + \frac{1}{e}.[g(\nabla_u \xi, u)]^2 + \\ + g({}_{rel}a, \xi_\perp) + \frac{l}{e}.g(\nabla_u \nabla_u \xi, u) \ .\end{gathered} \tag{106}$$

The last expression can help us to investigate the behavior of the length of a contravariant vector filed ξ when transported along a non-null contravariant vector field u. There are several physical interpretation of such type of a transport. If we interpret the vector field u as a time like vector field and as the 4-velocity of an observer in a (L_n, g)-space considered as a model of the space-time, then $u(ul_\xi) = (d^2 l_\xi)/(ds^2)$ represents the acceleration acting on the length l_ξ of the vector ξ along the observers trajectory $x^i(s)$. The vector field u can also be considered as a space like vector field with different from the above physical interpretation.

Special case: $g(u, \xi) = l := 0 : \xi = \xi_\perp$.

$$\begin{gathered}\pm\, l_{\xi_\perp}.(u(ul_{\xi_\perp})) = \mp(ul_{\xi_\perp})^2 + \frac{1}{2}.({}^s\nabla_u {}^s\nabla_u g)(\xi_\perp, \xi_\perp) + \\ + \frac{1}{n}.Q_u.[({}^s\nabla_u g)(\xi_\perp, \xi_\perp) + 2.g({}_{rel}v, \xi_\perp)] + \frac{1}{2n}.(uQ_u).l^2_{\xi_\perp} + \\ +\frac{2}{e}.({}^s\nabla_u g)(u, \xi_\perp).g(\nabla_u \xi_\perp, u) + 2.({}^s\nabla_u g)({}_{rel}v, \xi_\perp) + \\ +g({}_{rel}v, {}_{rel}v) + \frac{1}{e}.[g(\nabla_u \xi_\perp, u)]^2 + \ g({}_{rel}a, \xi_\perp) \ .\end{gathered} \tag{107}$$

Special case: $g(u,\xi) = l := 0 : \xi = \xi_\perp$, $\nabla_\xi g = 0$ for $\forall \xi \in T(M)$ $[g_{ij;k} := 0]$ (V_n- or U_n-spaces).

$$\pm\ l_{\xi_\perp}.(u(ul_{\xi_\perp})) = \mp(ul_{\xi_\perp})^2 + g({}_{rel}v,{}_{rel}v) + \frac{1}{e}.[g(\nabla_u\xi_\perp, u)]^2 + \ g({}_{rel}a, \xi_\perp)\ . \tag{108}$$

Special case: ${}^s\nabla_u g := 0 : \nabla_u g = \frac{1}{n}.Q_u.g$ (Weyl's space with torsion).

$$\begin{array}{c} \pm l_\xi.(u(ul_\xi)) = \mp(ul_\xi)^2 + \frac{2}{n}.Q_u.[g({}_{rel}v, \xi_\perp) + \frac{l}{e}.g(\nabla_u\xi, u)]+ \\ +\frac{1}{2n}.(uQ_u).(l^2_{\xi_\perp} + \frac{l^2}{e}) + g({}_{rel}v,{}_{rel}v) + \frac{1}{e}.[g(\nabla_u\xi, u)]^2 + g({}_{rel}a, \xi_\perp)+ \\ +\frac{l}{e}.g(\nabla_u\nabla_u\xi, u)\ . \end{array} \tag{109}$$

Special case: ${}^s\nabla_u g := 0 : \nabla_u g = \frac{1}{n}.Q_u.g$, $\quad g(u,\xi) = l := 0$, $\quad g(u,\eta) = \widetilde{l} := 0$ (Weyl's space with torsion and orthogonal to u vector fields $\xi_\perp$ and $\eta_\perp$).

$$\begin{array}{c} \pm l_{\xi_\perp}.(u(ul_{\xi_\perp})) = \mp(ul_{\xi_\perp})^2 + \frac{2}{n}.Q_u.g({}_{rel}v, \xi_\perp)+ \\ +\frac{1}{2n}.(uQ_u).l^2_{\xi_\perp} + g({}_{rel}v,{}_{rel}v) + \frac{1}{e}.[g(\nabla_u\xi_\perp, u)]^2 + g({}_{rel}a, \xi_\perp)\ . \end{array} \tag{110}$$

The representations of ul_ξ, $u(ul_\xi)$ and $u[\cos(\xi,\eta)]$ can be useful tools for the considerations of the motion of physical systems with given dimensions in (L_n, g)-spaces. The induced by the torsion kinematic characteristics of the relative velocity ${}_{rel}v$ and the relative acceleration ${}_{rel}a$ have an equal position to the other kinematic characteristics induced by the curvature or by external forces.

The relative acceleration between two (neighbor) points (particles) in an U_n-space ($n = 4$) is also related to quantities induced by non-autoparallel motions (under external forces), by the curvature and by the torsion.

3.4 Rate of change of the rate of change of the length of the vector field $\xi_\perp$ over an U_n-space

From the relation in U_n-spaces

$$ul_{\xi_\perp} = \pm\frac{1}{l_{\xi_\perp}}.g(\nabla_u\xi_\perp, \xi_\perp)\ , \quad l_{\xi_\perp} \neq 0\ ,$$

after covariant differentiation along the vector field u, we obtain for U_n-spaces the relation

$$u(ul_{\xi_\perp}) = \mp\frac{1}{l^2_{\xi_\perp}}.(ul_{\xi_\perp}).g(\nabla_u\xi_\perp, \xi_\perp)\pm\frac{1}{l_{\xi_\perp}}.g(\nabla_u\nabla_u\xi_\perp, \xi_\perp)\pm\frac{1}{l_{\xi_\perp}}.g(\nabla_u\xi_\perp, \nabla_u\xi_\perp)\ ,$$

where

$$\mp\frac{1}{l_{\xi_\perp}}.g(\nabla_u\xi_\perp,\xi_\perp)=\mp ul_{\xi_\perp}\ ,\ \ \mp\frac{1}{l^2_{\xi_\perp}}.(ul_{\xi_\perp}).g(\nabla_u\xi_\perp,\xi_\perp)=\mp\frac{1}{l^3_{\xi_\perp}}.[g(_{rel}v,\xi_\perp)]^2\ .$$

By the use of the decomposition of $\nabla_u\nabla_u\xi_\perp$,

$$\nabla_u\nabla_u\xi_\perp=\frac{\widehat{l}}{e}.u+\,_{rel}a\ ,\quad \widehat{l}=g(\nabla_u\nabla_u\xi_\perp,u)\ ,$$

under the condition $g(u,\xi_\perp)=l=0$, we obtain

$$u(ul_{\xi_\perp})=\mp\frac{1}{l^3_{\xi_\perp}}.[g(_{rel}v,\xi_\perp)]^2\pm\frac{1}{l_{\xi_\perp}}.g(_{rel}a,\xi_\perp)\pm\frac{1}{l_{\xi_\perp}}.\frac{\overline{l}\,^2}{e}\pm\frac{1}{l_{\xi_\perp}}g(_{rel}v,_{rel}v)\ , \tag{111}$$

where

$$\overline{l}=g(\nabla_u\xi_\perp,u)=\nabla_u[g(\xi_\perp,u)]-(\nabla_u g)(\xi_\perp,u)-g(\xi_\perp,\nabla_u u)\ .$$

From the last relation, it follows for U_n-spaces ($\nabla_u g=0$) and for $l=g(u,\xi_\perp)=0$ that

$$\overline{l}=g(\nabla_u\xi_\perp,u)=-g(\xi_\perp,\nabla_u u)=-g(\xi_\perp,a)\ ,\quad \nabla_u u=a\ .$$

For (L_n,g)-spaces and $l=0$, it follows that

$$\overline{l}=g(\nabla_u\xi_\perp,u)=-(\nabla_u g)(\xi_\perp,u)-g(\xi_\perp,\nabla_u u)\ .$$

Then we obtain for U_n-spaces

$$u(ul_{\xi_\perp})=\pm\frac{1}{l_{\xi_\perp}}.\{g(_{rel}a,\xi_\perp)-\frac{1}{l^2_{\xi_\perp}}[g(_{rel}v,\xi_\perp)]^2+\frac{1}{e}.[g(\xi_\perp,a)]^2+g(_{rel}v,_{rel}v)\}\ . \tag{112}$$

Therefore, for $u=\frac{d}{ds}$ the relation follows

$$\frac{d^2l_{\xi_\perp}}{ds^2}=\pm\frac{1}{l_{\xi_\perp}}.\{g(_{rel}a,\xi_\perp)+g(_{rel}v,_{rel}v)+\frac{1}{e}.[g(a,\xi_\perp)]^2-\frac{1}{l^2_{\xi_\perp}}[g(_{rel}v,\xi_\perp)]^2\}\ . \tag{113}$$

If $\pounds_u\xi_\perp=0$, then $_{rel}a=\overline{g}[A(\xi_\perp)]$, $_{rel}v=\overline{g}[d(\xi_\perp)]$. It is obvious that the existing kinematic terms induced by the torsion in A and in d could compensate the action of the kinematic terms induced by the curvature or by external forces in such a way that no change of rate of the rate of change of $l_{\xi_\perp}$ could be registered. If $_{rel}a=0$, $_{rel}v=0$ and $g(a,\xi_\perp)=0$, then $(d^2l_{\xi_\perp}/ds^2)=0$.

4 Conclusion

The recent development of the mathematical models of the space-time shows the possible use of spaces with affine connections and metrics. Such type of spaces have the torsion as an intrinsic characteristic. Even in (pseudo) Riemannian spaces with torsion (U_n-spaces) the tensor of the torsion and the kinematic characteristics related to it could influence under certain conditions the effects in the space-time caused by the curvature tensor and by external forces. Especially, the torsion could lead up to possible new gravitational experiments. In such cases a consideration of the influence of the torsion as a characteristic of the space-time appears as a necessary step to a better understanding of the properties of the space-time related to the gravitational interaction between physical systems.

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